Given: $A \le f(a+)$ and $B \ge f(b-)$

we can redefine f at the end points a and b

Such that

$f(a) = A$ and $f(b) = B$ $\longrightarrow$ (3)

The modified f is still increasing on $[a,b]$.

Also changing the value of f at a finite number of points does not affect the value of the Riemann integral.

Using (3) in (2), we have,

$$\int_a^b f(x)\, g(x)\, dx = A\int_a^{x_0} g(x)\, dx + B\int_{x_0}^{b} g(x)\, dx \rightarrow (4)$$

To prove (ii) Bonnet's theorem,

TPT $\int_a^b f(x)\, g(x)\, dx = B\int_{x_0}^{b} g(x)\, dx$

where $x_0 \in [a,b]$

Given, $f(x) \ge 0 \ \forall\ x \in [a,b]$

$\therefore f(a+) = \lim_{x \to a} f(a) \ge 0$

$\Rightarrow f(a+) \ge 0$

If we define $A=0$ then $A = f(a+)$ satisfied.

Substitute $A=0$ in (4) we have,

$$\lim_{y \to y_0} \int_a^b \cancel{f(x,y)\, d\alpha(x)} = \int_a^b \lim_{y \to y_0} \cancel{f(x,y)\, d\alpha(x)}$$

$$= \cancel{\int_a^b f(x, y_0)\, d\alpha(x)}$$

$$\int_a^b f(x)\, g(x)\, dx = B\int_{x_0}^{b} g(x)\, dx$$

Hence proved.

Theorem : 13

R.S.I Depending on a parameter:

Let f be continuous at each point (x,y) of a rectangle
$Q = \{(x,y) : a \le x \le b \,;\, c \le y \le d\}$

Assume that α is of bounded variation on $[a,b]$ and let 'F' be the function defined on $[c,d]$ by the equation

$$F(y) = \int_a^b f(x,y)\, d\alpha(x)$$

Then F is continuous on $[c,d]$. In other words if $y_0 \in [c,d]$ we have

$$\lim_{y \to y_0} \int_a^b f(x,y)\, d\alpha(x) = \int_a^b \lim_{y \to y_0} f(x,y)\, d\alpha(x)$$

$$= \int_a^b f(x, y_0)\, d\alpha(x)$$

Proof:

Given that : (i) f is continuous at each point (x,y) of a rectangle.

$$Q = \{(x,y) : a \le x \le b \,:\, c \le y \le d\}$$

(ii) α is of bounded variation on $[a,b]$

(iii) $F(y) = \int_a^b f(x,y)\, d\alpha(x)$, $\forall\, y \in [c,d]$

To prove that : F is continuous on $[c,d]$

(ie) If $y_0 \in [c,d]$ then to prove that

$$\lim_{y \to y_0} F(y) = F(y_0)$$

$\therefore$ It suffices to prove the theorem when α increasing on $[a,b]$.

Since f is continuous on compact set Q, f is uniformly continuous.

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(ie) For any given $\epsilon > 0$, there exists $\delta > 0$ (depending only on ϵ) such that for every points $z = (x, y)$ and $z' = (x', y'_0)$ in Q with

$|z - z'| < \delta$ we have

$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

$$|f(x,y) - f(x', y')| < \epsilon / \alpha(b) - \alpha(a) \longrightarrow (1)$$

Let $y' \in [c, d]$ such that $|y - y'| < \delta$ then we have

likewise

$$|F(y) - F(y'_0)| = \left| \int_a^b f(x,y)\, d\alpha(x) - \int_a^b f(x, y'_0)\, d\alpha(x) \right|$$

$$= \left| \int_a^b [f(x,y) - f(x, y'_0)]\, d\alpha(x) \right|$$

$$\leq \int_a^b |f(x,y) - f(x, y'_0)|\, d\alpha(x)$$

$[\because \alpha \text{ is increasing} \Rightarrow |\alpha(x)| = \alpha(x)]$

$$< \int_a^b \frac{\epsilon}{\alpha(b) - \alpha(a)}\, d\alpha(x)$$

$$= \frac{\epsilon}{\alpha(b) - \alpha(a)} [\alpha(x)]_a^b$$

$$= \frac{\epsilon}{\cancel{\alpha(b) - \alpha(a)}} [\cancel{\alpha(b) - \alpha(a)}] = \epsilon$$

$\Rightarrow |F(y) - F(y'_0)| < \epsilon$ whenever $|y - y'| < \delta$

$\Rightarrow F$ is continuous at $y'_0 \in [c, d]$

$\Rightarrow F$ is continuous on $[c, d]$

In other words, if $y_0 \in [c, d]$ we have

$$\lim_{y \to y_0} \int_a^b f(x,y)\, d\alpha(x) = \int_a^b \lim_{y \to y_0} f(x,y)\, d\alpha(x)$$

$$= \int_a^b f(x, y_0)\, d\alpha(x)$$

Hence the proof.

Theorem: 14

If f is continuous on the rectangle $[a,b] \times [c,d]$ and if $g \in R$ on $[a,b]$. then the function F defined by the equation

$$F(y) = \int_a^b g(x)\ f(x,y)\,dx$$ is continuous on $[a,b]$

i.e.) If $y_0 \in [c,d]$ we have,

$$\lim_{y \to y_0} \int_a^b g(x)\ f(x,y)\,dx = \int_a^b g(x)\ f(x,y_0)\,dx.$$

Proof: Let $G(x) = \int_a^x g(t)\,dt$, $x \in [a,b]$

Since $g \in R$ and $G(x) = \int_a^x g(t)\,dt$ on $[a,b]$.

if $\alpha(x) = x$ then $\alpha \uparrow$ on $[a,b]$, then by thm. 9 of unit 3, G is of bounded variation on $[a,b]$

Since f is continuous on $[a,b] \times [c,d]$, $f \in R$

Since $g \in R$, α is $\uparrow$ on $[a,b]$ and $f \in R$, and

$G(x) = \int_a^x g(t)\,dt$, By thm: 10 (unit 3)

$$\int_a^b f(x)\ g(x)\ d\alpha(x) = \int_a^b f(x)\ dG(x)$$

Hence, $$\int_a^b f(x,y) g(x)\,dx = \int_a^b f(x,y)\ dG(x)$$

$$\therefore \quad F(y) = \int_a^b f(x,y)\ dG(x).$$

Since f is continuous on $[a,b] \times [c,d]$ and

$F(y) = \int_a^b f(x,y)\ dG(x)$ on $[c,d]$ and G is of b.v on $[a,b]$, then by thm: 13 (unit 3), F is continuous on $[c,d]$.

Theorem : 15

Differentiation Under the Integral Sign:

Let $R = \{(x,y) : a \le x \le b ; c \le y \le d\}$

Assume that α is of bounded variation on $[a,b]$ and for each fixed y in $[c,d]$, assume that the integral $F(y) = \int_a^b f(x,y)\, d\alpha(x)$ exists. If the partial derivative $D_2 f$ is continuous on R the derivative $F'(y)$ exist for each y in $[c,d]$ and is given by $F'(y) = \int_a^b D_2 f(x,y)\, d\alpha(x)$.

Note:

In particular when $g \in R$ on $[c,d]$ and $\alpha(x) = \int_a^x g(t)\, dt$ we get $F(y) = \int_a^b f(x,y)\, g(x)\, dx$ and

$$F'(y) = \int_a^b D_2 f(x,y)\, g(x)\, dx.$$

Proof:

Given, i) $R = \{(x,y) : a \le x \le b : c \le y \le d\}$

ii) α is of b.v on $[a,b]$

iii) $F(y) = \int_a^b f(x,y)\, d\alpha(x)$, for each fixed $y \in [c,d]$

iv) partial derivative $D_2 f$ is continuous on

TP: $F'(y)$ exists for every $y \in (c,d)$

ie) $F'(y_0) = \lim_{y \to y_0} \dfrac{F(y) - F(y_0)}{y - y_0}$ exists, for Some $y_0 \in (c,d)$

If $y_0 \in (c,d)$ and $y \ne y_0$ we have

$$F(y) - F(y_0) = \int_a^b f(x,y)\, d\alpha(x) - \int_a^b f(x,y_0)\, d\alpha(x)$$

$$= \int_a^b [f(x,y) - f(x,y_0)]\, d\alpha(x)$$

$$\Rightarrow F(y) - F(y_0) = \int_a^b D_2 f(x,\bar{y})\,(y-y_0)\, d\alpha(x)$$

$$\Rightarrow \frac{F(y)-F(y_0)}{y-y_0} = \int_a^b D_2 f(x,\bar{y})\, d\alpha(x)$$

$\Big\{$ ∵ By Lagrange mean value then $\exists$ $\bar{y}$ such that $y < \bar{y} < y_0$ & $f'(x,\bar{y}) = \dfrac{f(x,y) - f(x,y_0)}{y-y_0}$

Taking limit as $y \to y_0$ on both sides we have,

$$\Rightarrow \lim_{y\to y_0} \frac{F(y) - F(y_0)}{y-y_0} = \lim_{y\to y_0} \int_a^b D_2 f(x,\bar{y})\, d\alpha(x)$$

$$\Rightarrow F'(y_0) = \int_a^b \lim_{y\to y_0} D_2 f(x,\bar{y})\, d\alpha(x)$$

$$\Rightarrow F'(y_0) = \int_a^b D_2 f(x,y_0)\, d\alpha(x)$$

$$\Rightarrow F'(y) = \int_a^b D_2 f(x,y)\, d\alpha(x), \quad \forall \ y \in [c,d].$$

Note:

In particular when $g \in R$ on $[a,b]$ and $\alpha(x) = \int_a^x g(t)\,dt$ by first fundamental theorem of integral calculus,

$\alpha'(x) = g(x)$

then we have, $F(y) = \int_a^b f(x,y)\, d\alpha(x)$

$$= \int_a^b f(x,y)\, \alpha'(x)\, dx$$

$$= \int_a^b f(x,y)\, g(x)\, dx$$

And $F'(y) = \int_a^b D_2 f(x,y)\, d\alpha(x)$

$$= \int_a^b D_2 f(x,y)\, \alpha'(x)\, dx$$

$$F'(y) = \int_a^b D_2 f(x,y)\, g(x)\, dx$$

Section : 7.25

Theorem : 16

Interchanging the order of Integration

Let $R = \{(x, y) : a \le x \le b ; c \le y \le d\}$

Assume that α is of bounded variation on $[a, b]$. β is of bounded variation on $[c, d]$ and f is continuous on R.

If $(x, y) \in R$ define $F(y) = \int_a^b f(x, y)\, d\alpha(x)$

$G(x) = \int_c^d f(x, y)\, d\beta(y)$, then $F \in R(\beta)$ on $[c, d]$ and $G \in R(\alpha)$ on $[a, b]$. And we have.

$$\int_c^d F(y)\, d\beta(y) = \int_a^b G(x)\, d\alpha(x)$$

In other words we may interchange the order of integration as follows.

$$\int_c^d F(y)\, d\beta(y) = \int_a^b G(x)\, d\alpha(x)$$

$$\int_a^b \left[\int_c^d f(x, y)\, d\beta(y) \right] d\alpha(x) = \int_c^d \left[\int_a^b f(x, y)\, d\alpha(x) \right] d\beta(y)$$

Proof :

Given,

(i) $R = \{(x, y) : a \le x \le b : c \le y \le d\}$

(ii) α is of b.v on $[a, b]$

(iii) β is of b.v on $[c, d]$

(iv) f is continuous on R.

(v) $F(y) = \int_a^b f(x, y)\, d\alpha(x)$ and

$G(x) = \int_c^d f(x, y)\, d\beta(y)$

To prove that : $F \in R(\beta)$ on $[c, d]$

Since f is continuous on R and

$F(y) = \int_a^b f(x,y)\, d\alpha(x)$ and α is of b.v on $[a,b]$

then by thm: 13

F is continuous on $[c,d]$

Given: β is of b.v on $[c,d]$ and we have F is continuous on $[c,d]$

$F \in R(\beta)$ on $[c,d]$ (by thm 4)

$\|^{ly}$. f is continuous on Q and

$G_1(x) = \int_c^d f(x,y)\, d\beta(y)$ and α is of b.v on $[a,b]$

G_1 is continuous on $[a,b]$ by thm 13

Since α is of b.v on $[a,b]$ and G_1 is continuous on $[a,b]$

then by thm 4

$G_1 \in R(\alpha)$ on $[a,b]$

To prove that: $\int_c^d F(y)\, d\beta(y) = \int_a^b G_1(x)\, d\alpha(x)$

Since f is continuous on compact set on Q,

f is uniformly continuous on Q.

Then for any given $\epsilon > 0$, $\exists\ \delta > 0$ (depending only on ϵ)

Such that for every pair of points $z = (x,y)$ and $z' = (x', y')$ in Q, we have

$|f(x,y) - f(x', y')| < \epsilon$ whenever $|z - z'| < \delta$ $\longrightarrow (*)$

Let us now ~~Subinterva~~ divide Q in to n^2 equal rectangles by subdividing $[a,b]$ and $[c,d]$ each into n - equal parts where 'n' is chosen so that

$$\frac{b-a}{n} < \frac{\epsilon}{\sqrt{2}} \quad \text{and} \quad \frac{d-c}{n} < \frac{\epsilon}{\sqrt{2}}$$

writing $x_k = a + \dfrac{k(b-a)}{n}$

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$$y_k = c + \frac{k(d-c)}{n}$$

for $k = 0, 1, 2, 3 \dots n$ we have

$$G(x) = \int_c^d f(x,y)\, d\beta(y) = \int_{c=y_0}^{y_1} f(x,y)\, d\beta(y) + \int_{y_1}^{y_2} f(x,y)\, d\beta(y) + \dots + \int_{y_{n-1}}^{y_n = d} f(x,y)\, d\beta(y)$$

$$\Rightarrow G(x) = \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+1}} f(x,y)\, d\beta(y) \longrightarrow (1)$$

Now,

$$\int_a^b G(x)\, d\alpha(x) = \int_{x_0=a}^{x_1} G(x)\, d\alpha(x) + \int_{x_1}^{x_2} G(x)\, d\alpha(x) + \dots + \int_{x_{n-1}}^{x_n = b} G(x)\, d\alpha(x)$$

$$= \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} G(x)\, d\alpha(x) \longrightarrow (2)$$

Using (1) in (2) we have.

$$\int_a^b G(x)\, d\alpha(x) = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} \left\{ \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+1}} f(x,y)\, d\beta(y) \right\} d\alpha(x)$$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \int_{x_k}^{x_{k+1}} \left\{ \int_{y_j}^{y_{j+1}} f(x,y)\, d\beta(y) \right\} d\alpha(x)$$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \int_{x_k}^{x_{k+1}} \left\{ f(x, y_j') \left[\beta(y_{j+1}) - \beta(y_j)\right] \right\} d\alpha(x)$$

where y_j' lies between y_j and y_{j+1} { By Ist mean value thm for R.S.I $\int_a^b f\,d\alpha = c[\alpha(b) - \alpha(a)]$ where $c = f(x_0)$, $x_0 \in (a,b)$, By thm }

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(x_k', y_j') \left[\beta(y_{j+1}) - \beta(y_j)\right] \left[\alpha(x_{k+1}) - \alpha(x_k)\right]$$

where x_k' lies between x_k & x_{k+1}

$$\therefore \int_a^b G(x)\, d\alpha(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(x_k', y_j') \left[\beta(y_{j+1}) - \beta(y_j)\right] \left[\alpha(x_{k+1}) - \alpha(x_k)\right] \longrightarrow (4)$$

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where $(x_k'', y_k'') \in R_{k,j}$

Since $z' = (x_k', y_j')$ and $z'' = (x_k'', y_j'') \in R_{k,j}$ and hence

If $|z' - z''| < \delta$.

(4) $\Rightarrow |f(x_k', y_j') - f(x_k'', y_j'')| < \epsilon$

(2) & (3) $\Rightarrow \left| \int_a^b G(x)\, d\alpha(x) - \int_c^d F(x)\, d\beta(y) \right|$

$= \left| \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \{ f(x_k', y_j') - f(x_k'', y_j'') \} [\beta(y_{j+1}) - \beta(y_j)] [\alpha(x_{k+1}) - \alpha(x_k)] \right|$

$\Rightarrow \left| \int_a^b G(x)\, d\alpha(x) - \int_c^d f(x)\, d\beta(y) \right|$

$\leq \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} |f(x_k', y_j') - f(x_k'', y_j'')| [\beta(y_{j+1}) - \beta(y_j)] [\alpha(x_{k+1}) - \alpha(x_k)]$

$< \epsilon \sum_{k=0}^{n-1} \{\alpha(x_{k+1}) - \alpha(x_k)\} \sum_{j=0}^{n-1} \{\beta(y_{j+1}) - \beta(y_j)\}$

$< \epsilon [\alpha(b) - \alpha(a)] [\beta(d) - \beta(c)]$

$\Rightarrow \left| \int_a^b G(x)\, d\alpha(x) - \int_c^d F(x)\, d\beta(y) \right| < \epsilon [\alpha(b) - \alpha(a)] [\beta(d) - \beta(c)]$

Since ϵ is arbitrary,

$\left| \int_a^b G(x)\, d\alpha(x) - \int_c^d F(x)\, d\beta(y) \right| = 0$

$\Rightarrow \int_a^b G(x)\, d\alpha(x) - \int_c^d F(x)\, d\beta(y) = 0$

$\Rightarrow \int_a^b G(x)\, d\alpha(x) = \int_c^d F(x)\, d\beta(y)$

In other words

$$\int_a^b \left[\int_c^d f(x,y)\, d\beta(y) \right] d\alpha(x) = \int_c^d \left[\int_a^b f(x,y)\, d\alpha(x) \right] d\beta(y)$$

Hence the proof.